**Q1)** The Squire theorem, although one should be careful using it. Indeed, it stipulates that conclusions of the simplified analysis , without the transverse direction, applies to a full analysis with it, but for different non-dimensional numbers.

**Q2)** The proposed expansions are to be injected in Euler equations plus the temperature equation and collecting terms at order $\epsilon$.

**Q3)** It is the continuity equation, and, as written in (8), expresses the divergence of the perturbation field in velocity to be equal to zero. The physical counterpart is the incompressibility of the fluid (everything that arrives in a fluid control volume must go out : no fluid accumulation is possible).

**Q4)** The Boussinesq hypothesis is used. Namely, the density is assumed constant (of value $\rho_0$) in the continuity and momentum equations, excepted for the buoyancy term.

**Q5)** The flow is indeed said to be stably stratified as the heavy (cold) fluid is below the light (hot) fluid. No Rayleigh-Bénard instability is to be expected (which does not imply the flow to be stable)

**Q6)** We can get rid of the pressure term by taking the rotational of the momentum equation. Namely, we take the derivative of (5) with respect to $z$

$$(-i\omega + ikU)Du + ikU'u + U''v + U'Dv = -ikDp/\rho_0, \tag{1}$$

as well as the derivative of (6) with respect to $x$ (taking the latter derivative amounts to multiply the equation by $ik$)

$$(-i\omega + ikU)(ikv) = -ikDp/\rho_0 + ik\frac{\beta}{\rho_0}g\theta, \tag{2}$$

and subtract the former to the latter to obtain

$$(-i\omega + ikU)(Du - ikv) + ikU'u + U''v + U'Dv = -ik\frac{\beta}{\rho_0}g\theta \tag{3}$$

In terms of the streamfunction $v = -ik\psi$ and $u = D\psi$

$$\begin{aligned}(-i\omega + ikU)\left(D^2 - k^2\right)\psi + ikU'D\psi + U''(-ik\psi) + U'(-ikD\psi) = -ik\frac{\beta}{\rho_0}g\theta \Leftrightarrow \\ (-i\omega + ikU)\left(D^2 - k^2\right)\psi - ikU''\psi = -ik\frac{\beta}{\rho_0}g\theta\end{aligned} \tag{4}$$

but $\theta = -T'(z)v/(-i\omega + ikU)$, therefore

$$\begin{aligned}(-i\omega + ikU)\left(D^2 - k^2\right)\psi - ikU''\psi &= -ik\frac{\beta}{\rho_0}g\left(\frac{T'(z)ik\psi}{-i\omega + ikU}\right) \Leftrightarrow \\ (-i\omega + ikU)\left(D^2 - k^2\right)\psi - ikU''\psi &= \frac{\beta}{\rho_0}g\left(\frac{k^2T'(z)\psi}{-i\omega + ikU}\right) \Leftrightarrow \\ (-i\omega + ikU)\left(D^2 - k^2\right)\psi - ikU''\psi &= \left(\frac{k^2N^2}{-i\omega + ikU}\right)\psi\end{aligned} \tag{5}$$

with

$$N^2 = \frac{\beta g T'(z)}{\rho_0} \tag{6}$$

**Q7)** $[\beta] = [\rho_0]/[T]$, so

$$\left[N^2\right] = \frac{[\rho_0]}{[T]} * \frac{m}{s^2} * \frac{[T]}{m} * \frac{1}{[\rho_0]} = \frac{1}{s^2} \tag{7}$$

$N$ being in $s^{-1}$ (Hertz), it is indeed a frequency. Without stratification, the Taylor-Goldstein equation reduces to the Rayleigh equation.

**Q8)** It is a polynomial eigenvalue problem in a temporal sense (i.e. considering $\omega(k)$), and also in a spatial sense (i.e considering $k(\omega)$). Indeed there are terms proportional to $\omega$ and $\omega^2$.

**Q9)** With the chosen form for the Fourier mode $\propto e^{i(kx-\omega t)} = e^{i(kx-\omega_r t - i\omega_i t)} = e^{\omega_i t} e^{i(kx-\omega_r t)}$. Thus instability happens whenever $\omega_i > 0$

**Q10)** Let us first notice that $-i\omega$ becomes $-i(\omega_r + i\omega_i) = \omega_i - i\omega_r$. The Taylor-Goldstein becomes in turns :

$$
\begin{aligned}
&\left[\omega_i + i\left(kU - \omega_r\right)\right]\left(D^2 - k^2\right)\psi - ikU''\psi = \left(\frac{k^2N^2}{\left[\omega_i + i\left(kU-\omega_r\right)\right]}\right)\psi \Leftrightarrow \\
&\left[\omega_i + i\left(kU - \omega_r\right)\right]\left(D^2 - k^2\right)\psi - ikU''\psi = \left(\frac{k^2N^2\left[\omega_i - i\left(kU-\omega_r\right)\right]}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]}\right)\psi \Leftrightarrow \\
&\left(D^2 - k^2\right)\psi - \frac{ikU''}{\left[\omega_i + i\left(kU-\omega_r\right)\right]}\psi = \left(\frac{k^2N^2\left[\omega_i - i\left(kU-\omega_r\right)\right]}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]\left[\omega_i + i\left(kU-\omega_r\right)\right]}\right)\psi \Leftrightarrow \\
&\left(D^2 - k^2\right)\psi - \frac{ikU''}{\left[\omega_i + i\left(kU-\omega_r\right)\right]}\psi = \left(\frac{k^2N^2\left[\omega_i - i\left(kU-\omega_r\right)\right]^2}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]^2}\right)\psi
\end{aligned}
\tag{8}
$$

We multiply by $\psi^*$ (the complex conjugate of $\psi$) then integrate over $z$ from $z_a$ to $z_b$.

$$
\int \psi^*\left(D^2 - k^2\right)\psi dz - ik\int \frac{U''\psi\psi^*}{\left[\omega_i + i\left(kU-\omega_r\right)\right]}dz = \int\left(\frac{k^2N^2\left[\omega_i - i\left(kU-\omega_r\right)\right]^2}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]^2}\right)\psi\psi^* dz \tag{9}
$$

But notice that

$$
\int \psi^* D^2\psi dz = \cancel{\left[\psi^* D\psi\right]_{z_a}^{z_b}} - \int D\psi^* D\psi dz = -\int |D\psi|^2 dz \tag{10}
$$

where the cancellation of the boundary terms come from the no-slip boundary conditions. In this manner, (9) becomes

$$
-\int |D\psi|^2 + k^2|\psi|^2 dz - ik\int \frac{U''|\psi|^2}{\left[\omega_i + i\left(kU-\omega_r\right)\right]}dz = \int\left(\frac{k^2N^2\left[\omega_i - i\left(kU-\omega_r\right)\right]^2}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]^2}\right)|\psi|^2 dz \tag{11}
$$

We use $\left[\omega_i - i\left(kU-\omega_r\right)\right]^2 = \omega_i^2 - 2i\omega_i\left(kU-\omega_r\right) - \left(kU-\omega_r\right)^2$, and take the imaginary part of (11)

$$
\begin{aligned}
&-k\omega_i\int \frac{U''|\psi|^2}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]}dz = \int\left(\frac{-k^2N^2 2\omega_i\left(kU-\omega_r\right)}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]^2}\right)|\psi|^2 dz \\
&\omega_i\int \frac{U''|\psi|^2}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]}dz = \omega_i\int\left(\frac{2kN^2\left(kU-\omega_r\right)}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]^2}\right)|\psi|^2 dz \\
&\omega_i\int\left(U'' - \frac{2kN^2\left(kU-\omega_r\right)}{\omega_i^2 + \left(kU-\omega_r\right)^2}\right)\frac{|\psi|^2}{\omega_i^2 + \left(kU-\omega_r\right)^2}dz = 0
\end{aligned}
\tag{12}
$$

Since $|\psi|^2/[\omega_i^2 + (kU-\omega_r)^2]$ can't be negative, (12) is satisfied if $\omega_i = 0$ (no instability) or if there exist some $z \in ]z_a; z_b[$ where

$$
U'' \leq \frac{2kN^2\left(kU-\omega_r\right)}{\left[\omega_i^2 + \left(kU-\omega_r\right)^2\right]} \tag{13}
$$

which therefore constitutes a necessary condition for instability. Unfortunately, this equation is not a criterion of the flow itself, unlike Rayleigh's one, since it depends on the eigenvalue.

**Q11)** When a computation appears complicated, take your time, and divide it in easier sub-problems.

$$D\chi = -\frac{ikU'}{2}(-i\omega + ikU)^{-3/2}\psi + \frac{D\psi}{\sqrt{-i\omega + ikU}}, \tag{14}$$

so

$$(-i\omega + ikU)D\chi = -\frac{ikU'}{2\sqrt{-i\omega + ikU}}\psi + D\psi\sqrt{-i\omega + ikU}, \tag{15}$$

implying in turns

$$\begin{aligned}
D[(-i\omega + ikU)D\chi] &= -\frac{ik}{2}\left(\frac{U''}{\sqrt{-i\omega + ikU}} - \frac{ikU'^2}{2}(-i\omega + ikU)^{-3/2}\right)\psi - \frac{ikU'}{2\sqrt{-i\omega + ikU}}D\psi + \\
&\sqrt{-i\omega + ikU}D^2\psi + \frac{ikU'}{2\sqrt{-i\omega + ikU}}D\psi \\
&= -\frac{ik}{2}\left(\frac{U''}{\sqrt{-i\omega + ikU}} - \frac{ikU'^2}{2}(-i\omega + ikU)^{-3/2}\right)\psi + \sqrt{-i\omega + ikU}D^2\psi \\
&= -\frac{ik}{2}\left(\frac{U''}{\sqrt{-i\omega + ikU}} - \frac{ikU'^2}{2}(-i\omega + ikU)^{-3/2}\right)\psi \\
&+ \sqrt{-i\omega + ikU}k^2\psi + \frac{ikU''\psi}{\sqrt{-i\omega + ikU}} + \frac{k^2N^2}{(-i\omega + ikU)^{3/2}}\psi \\
&= \left[-\frac{ik}{2}U'' - \frac{k^2U'^2}{4(-i\omega + ikU)} + (-i\omega + ikU)k^2 + ikU'' + \frac{k^2N^2}{-i\omega + ikU}\right]\frac{\psi}{\sqrt{-i\omega + ikU}} \\
&= \left[\frac{ik}{2}U'' + k^2\frac{N^2 - U'^2/4}{(-i\omega + ikU)} + k^2(-i\omega + ikU)\right]\frac{\psi}{\sqrt{-i\omega + ikU}},
\end{aligned} \tag{16}$$

which is the desired equality.

**Q12)** Since

$$\begin{aligned}
\int D[(-i\omega + ikU)D\chi]\chi^* dz &= [\chi^*(-i\omega + ikU)D\chi]_{z_a}^{z_b} - \int D\chi^*(-i\omega + ikU)D\chi dz \\
&= -\int(-i\omega + ikU)|D\chi|^2 dz,
\end{aligned} \tag{17}$$

we have

$$-\int(\omega_i + i(kU - \omega_r))|D\chi|^2 dz = \int\left(k^2\left(N^2 - U'^2/4\right)\frac{[\omega_i - i\,(kU - \omega_r)]}{\left[\omega_i^2 + (kU - \omega_r)^2\right]} + \frac{ik}{2}U'' + k^2\,[\omega_i + i\,(kU - \omega_r)]\right)|\chi|^2 dz \tag{18}$$

If we now take the real part of (18), we are left with

$$\begin{aligned}
-\omega_i\int|D\chi|^2 dz - \int\left(\frac{\omega_i k^2\left(N^2 - U'^2/4\right)}{\left[\omega_i^2 + (kU - \omega_r)^2\right]} + k^2\omega_i\right)|\chi|^2 dz = 0 \Leftrightarrow \\
\omega_i\int\frac{k^2\left(U'^2/4 - N^2\right)}{\left[\omega_i^2 + (kU - \omega_r)^2\right]}|\chi|^2 dz = \omega_i\int|D\chi|^2 + k^2|\chi|^2 dz.
\end{aligned} \tag{19}$$

but $\left[\omega_i^2 + (kU - \omega_r)^2\right] > 0$. If $U^2/4 - N^2 < 0$ everywhere in the flow domain, then we equate a positive and a negative number and the only solution is $\omega_i = 0$, so no instability. Therefore, the necessary condition for an instability to occur is that there exist a pocket in $y$ where $U'^2/4 + N^2 > 0$.

**Q13)** A necessary condition for the instability to occur is :

$$\frac{N^2(z)}{U'(z)^2} = \frac{\beta g T'(z)}{\rho_0 U'(z)^2} < \frac{1}{4}$$

to be verified somewhere in the flow. It is clear that a larger $\beta$ and or $T'(z)$ makes it less likely to occur (everything else being fixed), thus increasing the strength of stratification stabilizes the flow. Notice that the dimensionless number $Ri$ compare the potential energy of the flow $\beta g T'(z) L^2$ over the shear-induced kinetic one $\rho_0 U'(z)^2 L^2$ (where $L$ is the length scale of the problem). As $Ri \to 0$, the density field is easily disturbed by the velocity one, stratification not being strong enough to resist the shear.

**Q14)** Integrating

$$N^2(z) = J\left(1 - \tanh^2(Rz)\right) = \frac{\beta g T'(z)}{\rho_0}$$

leads to $T(z) = T_a + \frac{\rho_0 J}{\beta g}\left(\frac{\tanh(Rz)}{R} - \frac{\tanh(Rz_a)}{R}\right)$. This is an increasing function of $z$, therefore the hotter fluid is above thus the flow is said to be "stably stratified".

**Q15)** $N-1$ interior points and $2(N-1)$ eigenvalues.

**Q16)** For $R = 1$ :

$$Ri(z) = \frac{N(z)^2}{U'(z)^2} = \frac{J}{\left(1 - \tanh^2(z)\right)}$$

Its minimum value is equal to $J$, for the denominator is at most equal to one (in $z = 0$)

**Q17)** For $R = 1$, the necessary condition for the instability to occur is $J < 1/4$. In other terms, no instability can occur if $J > 1/4$, which seems to be the case on figure $3a$.

**Q18)** The instability necessary (not sufficient) condition predicts that the flow could be unstable for all possible $J$. Figure 4 shows that unstable modes indeed exist for all the $J$ considered.

**Q19)** Increasing the stratification suppresses the Kelvin-Helmoltz instability for that the entertainment of the heavy fluid in the Kelvin-Helmotz billows cost too much potential energy.

Increasing the stratification creates the Holmboe instability, briefly make its growth rate higher then suppresses it for high $J$. The most unstable wavenumber seems to monotonously increase with $J$. Everything else being fixed, a sharp density (temperature) interface (strong $R$ ) seems necessary to create the Holmboe instability.

**Q20)** For Kelvin-Helmoltz instability : the dispersion relation is said to be non-dispersive, as the frequency does not change with the wavenumber. Thus $d\omega/dk = 0 \forall k$ meaning that the wavepacket is static (it growth on place) : the instability is certainly absolute.

For Holmboe instability, it is impossible to deduce if the instability will be convective or absolute be visual inspection only.